Lecture 6: Separatedness & Valuation Criterion

Note Title 9/1/2019

Schemes w/ Zariski topology are NOT Hausdorff.

Definition: $f: X \to Y$ morphism of schemes

- $$\begin{tikzcd} X \arrow[dr, "\exists\Delta"] \arrow[drr, bend left, "id"] \arrow[ddr, bend right, "id"'] & & \\ & X \times_Y X \arrow[r, "p_1"] \arrow[d, "p_2"] & X \arrow[d, "f"] \\ & X \arrow[r, "f"] & Y \end{tikzcd}$$

  Δ: diagonal morphism

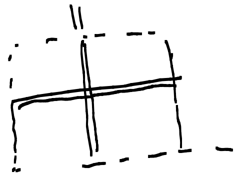
- f is <u>separated</u> if Δ is an closed immersion.

  or X is seperated over Y

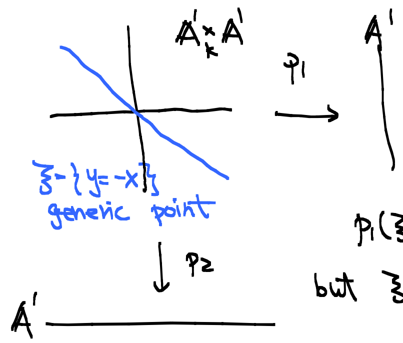
ex. $X =$  $\mathbb{A}^1_k$

$X \xrightarrow{\Delta} X \times_k X$ is NOT seperated

$\parallel$



ex. $\Delta(X)$ is NOT the set-theoretical diagonal



$p_1(\xi) = p_2(\xi)$

but $\xi \notin \Delta(X)$

Lemma 1: Morphisms between affine schemes are separated.

thus separatedness is NOT a local condition

pf: $$\begin{tikzcd} \mathrm{Spec}B \arrow[r, "f"] \arrow[d, equal] & \mathrm{Spec}A \arrow[d, equal] \\ X & Y \end{tikzcd}$$

$$\begin{tikzcd} X \arrow[r] \arrow[d, equal] & X \times_Y X \arrow[d, equal] \\ \mathrm{Spec}B \arrow[r, hook] & \mathrm{Spec}(B \otimes_A B) \end{tikzcd} \approx B \otimes_A B \twoheadrightarrow B$$

Lemma 2: $f: X \to Y$ is seperated iff $\Delta(X) \underset{\text{closed}}{\subseteq} X \times_Y X$

pf: $(\Rightarrow)$ is obvious.

($\Leftarrow$) Need to show that 1. $X \xrightarrow[\text{top}]{\simeq} \Delta(X)$

2. $\mathcal{O}_{X\times_Y X} \twoheadrightarrow \Delta_* \mathcal{O}_X$

1.
$$\begin{tikzcd}
X \arrow[dr, "\Delta"] \arrow[ddr, bend right, "id"'] & & \\
& X\times_Y X \arrow[r] \arrow[d] & X \arrow[d] \\
& X \arrow[r] & Y
\end{tikzcd}$$

$\Rightarrow X \xrightarrow{\simeq} \Delta(X)$

2, can be checked locally and follows from Lemma 1.

Theorem (Valuation Criterion of Separatedness)

$f: X \longrightarrow Y$, then TFAE
Notherian

1. f is seperated
2. For any K = field, R = valuation ring w/ quotient field K

$$\begin{tikzcd}
\mathrm{Spec}\,K = U \arrow[r] \arrow[d, "i"'] & X \arrow[d] \\
\mathrm{Spec}\,R = T \arrow[r] \arrow[ur, dotted] & Y
\end{tikzcd}$$

$\exists$ at most one such lifting

" A morphism defined on a punctured disc has at most one extension "

Examples:

1. Open/closed immersions are seperated. Lemma 2
2. Composition of separated morphisms are separated.
3. Separated morphisms are stable under base change.
4. $f: X \longrightarrow Y$ separated over S
$f': X' \longrightarrow Y'$

then $f \times f' : X \times_S X' \longrightarrow Y \times_S Y'$ separated

5. $X \xrightarrow{f} Y \xrightarrow{g} Z$, $g \circ f$ separated $\Longrightarrow$ f is separated

6. $f : X \longrightarrow Y$ separated   Lemma 2

iff $Y = \bigcup \mathcal{U}_i$ (open cover) s.t. $f^{-1}(\mathcal{U}_i) \longrightarrow \mathcal{U}_i$ separated, $\forall i$

pf:

2.
$$\begin{tikzcd} U \arrow[r, "\tilde{h}"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ & Y \arrow[d, "g"] \\ T \arrow[uur, shift left, "\rho_1"] \arrow[uur, shift right, "\rho_2"'] \arrow[ur, "f \circ \rho_i"] \arrow[r, "h"'] & Z \end{tikzcd}$$

Assume $\exists$ two lifting $\rho_1, \rho_2 : T \longrightarrow X$

g separated $\Longrightarrow$ $f \circ \rho_1 = f \circ \rho_2$

f separated $\Longrightarrow$ $\rho_1 = \rho_2$

5.
$$\begin{tikzcd} U \arrow[r, "\tilde{h}"] \arrow[d, "i"'] & X \arrow[d, "f"] \\ T \arrow[ur, shift left, "\rho_1"] \arrow[ur, shift right, "\rho_2"'] \arrow[r, "h"] \arrow[dr, "g \circ h"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

Lemma 3. With notation in the theorem

$U = \operatorname{Spec} K \longrightarrow X$ $\approx$ $x_1 \in X$, $k(x_1) \subseteq K$ (inclusion of fields)

$T = \operatorname{Spec} R \longrightarrow X$ $\approx$ $x_0, x_1 \in X$, $x_0 \in \overline{\{x_1\}}$

$k(x_1) \subseteq K$

$Z =$ reduced subscheme of $\overline{\{x_1\}}$

R dominates $\mathcal{O}_{x_0, Z}$

pf: $U \to X \Leftrightarrow \mathcal{O}_{x_1, X} \longrightarrow K$ local homomorphism

$\Leftrightarrow k(x_1) = \mathcal{O}_{x_1, X} / m_{x_1, X} \hookrightarrow K / 0$

$T = \operatorname{Spec} R$ has two points t_0, t_1 $\therefore \overline{\{t_1\}} \ni t_0$

$t_0 = m_R \supseteq (0) = t_1$

$U \to T \xrightarrow{h} X \iff x_0 = h(t_0),\ x_1 = h(t_1)$ $\quad k(x_1) \subseteq K$

reduced $T \dashrightarrow Z \to X$

$\mathcal{O}_{x_0, Z} \to R$ local homomorphism

w/ function field $k(x_1) \subseteq K$

Lemma 4. $f: X \to Y$ quasi-compact morphism of schemes

$\forall V \underset{\text{open}}{\subseteq} Y,\ f^{-1}(V)$ is quasi-compact

every open cover has a finite cover

then $f(X) \subseteq Y$ is closed iff $\left(\begin{array}{l} x_1 \in f(X),\ x_0 \in \overline{\{x_1\}} \\ \text{then } x_0 \in f(X) \end{array} \right)$

pf: ($\Longrightarrow$) obvious

($\Longleftarrow$) WLOG may assume $Y = \overline{f(X)}$ w/ reduced structure

and Goal: $y \in Y$, then $y \in f(X)$

reduction to commutative algebra

It suffices to check it locally, may assume Y affine

f: quasi-compact $\Longrightarrow$ $f^{-1}(Y)$ covered by finitely many affine charts X_i

replace X by X_i

may assume that X is affine

Now $X = \operatorname{Spec} A \xrightarrow{f} Y = \operatorname{Spec} B$ dominant $\iff$ $B \hookrightarrow A$

$y = \mathfrak{p} \in Y$, $\mathfrak{p} \triangleleft B$

$y' = \mathfrak{p}'$, $\mathfrak{p} \supseteq \mathfrak{p}'$ minimal ideal

intersection of prime ideals are primes

existence by Zorn's lemma.

$\overline{\{y'\}} \ni y$

It suffices to prove that $y' \in f(X)$.

$$(B \hookrightarrow A)_{\mathfrak{p}'} \Longrightarrow B_{\mathfrak{p}'} \hookrightarrow A \otimes B_{\mathfrak{p}'}$$

localization exact functor

$B_{\mathfrak{p}'}$ field

$\mathfrak{q}_0' \triangleleft A \otimes B_{\mathfrak{p}'}$ prime

local ring of a minimal prime is a field

$\mathfrak{q}_0' \cap B_{\mathfrak{p}'} = (0)$

$$\begin{tikzcd}
\overset{\mathfrak{p}'}{B} \arrow[r] \arrow[rrd, bend right] & \overset{(0)}{B_{\mathfrak{p}'}} \arrow[r, hook] & \overset{\mathfrak{q}_0'}{A \otimes B_{\mathfrak{p}'}} \\
& & A \ \mathfrak{q}' \arrow[u]
\end{tikzcd}$$

then $\mathfrak{q}' \cap B = \mathfrak{p}'$

i.e. $f(\mathfrak{q}') = \mathfrak{p}' = y' \in f(X)$

Proof of the valuation criterion:

($\Longrightarrow$) Assume $X \xrightarrow{\Delta} X \times_Y X$ closed embedding

$$\begin{tikzcd}
t_1 = U \arrow[r, "\tilde{h}"] \arrow[d, "i"'] & X \arrow[d, "f"] \\
\{t_0, t_1\} = T \arrow[r, "h"'] \arrow[ur, shift left, "\rho_1"] \arrow[ur, shift right, "\rho_2"'] & Y
\end{tikzcd}
\Longrightarrow
\begin{tikzcd}
T \arrow[dr, "\exists!\, h''"] \arrow[drr, bend left, "\rho_1"] \arrow[ddr, bend right, "\rho_2"'] & & \\
& X \times_Y X \arrow[r, "p_2"] \arrow[d, "p_1"] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

$\because f \circ \rho_1 = f \circ \rho_2 = h$

Claim:
$$\begin{tikzcd}
\boxed{T \arrow[r, "h''"] \arrow[d, dashed] & X \times_Y X} \\
X \arrow[ur, "\Delta"'] &
\end{tikzcd}$$
thus $\rho_1 = \rho_2$

$\rho_1 \circ i = \rho_2 \circ i = \tilde{h} \Longrightarrow \underset{x_1}{\rho_1(t_1) \shortparallel} = \rho_2(t_1)$ & $h''(t_1) \in \Delta(X)$

thus $h''(t_0) \in \overline{h''(t_1)} \subseteq \Delta(X)$

the two inclusions induced from ρ_1, ρ_2 coincide

$$\begin{array}{ccc} k(x_1) & \hookrightarrow & K \\ & & \cup \\ \mathcal{O}_{x_0, Z_1} & \longrightarrow & R \end{array}$$

($\Longleftarrow$) Lemma 2 $\Rightarrow$ suffices to check $\Delta(X) \underset{\text{closed}}{\subseteq} X \times_Y X$

X: Noetherian $\Rightarrow$ Δ quasi-compact

$\Longrightarrow$ suffices to check $\Delta(X)$ stable under specialization

Lemma 4 i.e. $\xi_1 \in \Delta(X)$, $\overline{\{\xi_1\}} \ni \xi_0 \Rightarrow \xi_0 \in \Delta(X)$

$K = k(\xi_1)$, $\mathcal{O} = \mathcal{O}_{\xi_0, \overline{\{\xi_1\}}} \subseteq K$
local ring

$\exists$ R valuation ring dominate $\mathcal{O}$

Lemma 3 $\Longrightarrow$

$$\begin{tikzcd}
 & & & X \arrow[dr, "f"] & \\
U \arrow[r] \arrow[urrr, bend left, "="] \arrow[drrr, bend right, "="'] & T \arrow[r, "h"] & X \times_Y X \arrow[ur, "p_1"] \arrow[dr, "p_2"'] & & Y \\
 & & & X \arrow[ur, "f"'] &
\end{tikzcd}$$

$t_1 \in U$, $\quad t_0, t_1 \in T \longmapsto \xi_0, \xi_1$

$$\begin{tikzcd}
U \arrow[r] \arrow[d, dotted] & X \times_Y X \\
X \arrow[ur, "\Delta"'] &
\end{tikzcd}$$

thus same as $T \to X \times_Y X$

unique lifting $\Longrightarrow$ $p_1 \circ h = p_2 \circ h$

$\therefore$ $h = \Delta$ & $\xi_0 \in \Delta(X)$

Remark: It suffices to check R = discrete valuation ring for valuation criterion of separatedness / properness.